

The bondage number of graphs on topological surfaces and Teschner's conjecture

Andrei Gagarin^{a,*}, Vadim Zverovich^b

^a*Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia,
B4P 2R6, Canada*

^b*Department of Mathematics and Statistics, University of the West of England, Bristol,
BS16 1QY, UK*

Abstract

The bondage number of a graph is the smallest number of its edges whose removal results in a graph having a larger domination number. We provide constant upper bounds for the bondage number of graphs on topological surfaces, improve upper bounds for the bondage number in terms of the maximum vertex degree and the orientable and non-orientable genera of the graph, and show tight lower bounds for the number of vertices of graphs 2-cell embeddable on topological surfaces of a given genus. Also, we provide stronger upper bounds for graphs with no triangles and graphs with the number of vertices larger than a certain threshold in terms of the graph genera. This settles Teschner's Conjecture in positive for almost all graphs.

Keywords: Bondage number, Domination number, Topological surface, Embedding on a surface, Euler's formula, Triangle-free graphs

1. Introduction

We consider simple finite non-empty graphs. For a graph G , its vertex and edge sets are denoted, respectively, by $V(G)$ and $E(G)$, $|V(G)| = n$ and $|E(G)| = m$. We also use the following standard notation: $d(v)$ for the degree of a vertex v in G , $\Delta = \Delta(G)$ for the maximum vertex degree of G , $\delta = \delta(G)$

*Corresponding author, fax: (1-902) 585-1074, phone: (1-902) 585-1419

Email addresses: andrei.gagarin@acadiau.ca (Andrei Gagarin),
vadim.zverovich@uwe.ac.uk (Vadim Zverovich)

for the minimum vertex degree of G , and $N(v)$ for the neighbourhood of a vertex v in G .

A set $D \subseteq V(G)$ is a *dominating set* if every vertex not in D is adjacent to at least one vertex in D . The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$. Clearly, for any spanning subgraph H of G , $\gamma(H) \geq \gamma(G)$. The *bondage number* of G , denoted by $b(G)$, is the minimum cardinality of a set of edges $B \subseteq E(G)$ such that $\gamma(G - B) > \gamma(G)$, where $V(G - B) = V(G)$ and $E(G - B) = E(G) \setminus B$. In a sense, the bondage number $b(G)$ measures integrity and reliability of the domination number $\gamma(G)$ with respect to the edge removal from G , which may correspond, e.g., to link failures in communication networks.

The bondage number was introduced by Bauer et al. [1] (see also Fink et al. [5]). Recently, it has been shown by Hu and Xu [11] that the decision problem for the bondage number is NP-hard. Also, they have conjectured that determining an actual set of edges corresponding to the bondage number is not even an NP-problem, which implies it is important to have any reasonable estimations and bounds on the bondage number in terms of other graph parameters and properties. Two unsolved classical conjectures for the bondage number of arbitrary and planar graphs are as follows.

Conjecture 1 (Teschner [19]). *For any graph G , $b(G) \leq \frac{3}{2}\Delta(G)$.*

Hartnell and Rall [9] and Teschner [20] showed that for the cartesian product $G_n = K_n \times K_n$, $n \geq 2$, the bound of Conjecture 1 is sharp, i.e. $b(G_n) = \frac{3}{2}\Delta(G_n)$. Teschner [19] also proved that Conjecture 1 holds when $\gamma(G) \leq 3$.

Conjecture 2 (Dunbar et al. [4]). *If G is a planar graph, then $b(G) \leq \Delta(G) + 1$.*

Trying to prove Conjecture 2, Kang and Yuan [15] have shown the following, with a simpler topological proof later discovered by Carlson and Develin [3].

Theorem 3 ([15, 3]). *For any connected planar graph G ,*

$$b(G) \leq \min\{8, \Delta(G) + 2\}.$$

This solves Conjecture 2 when $\Delta(G) \geq 7$, and Conjecture 1 for planar graphs with $\Delta(G) \geq 4$. Also, it is shown in [3] that $b(G) \leq \Delta(G) + 3$ for any

connected toroidal graph G , which solves Conjecture 1 for toroidal graphs with $\Delta(G) \geq 6$. In [8], we generalize this for any topological surface as follows.

Theorem 4 ([8]). *For a connected graph G of orientable genus h and non-orientable genus k ,*

$$b(G) \leq \min\{\Delta(G) + h + 2, \Delta(G) + k + 1\}. \quad (1)$$

Also, in [8], we indicate that the upper bound (1) can be improved for bigger values of the genera h and k by adjusting the proofs and should be helpful in solving Conjecture 1. Finally, we state the following general conjecture.

Conjecture 5 ([8]). *For a connected graph G of orientable genus h and non-orientable genus k , $b(G) \leq \min\{c_h, c'_k, \Delta(G) + o(h), \Delta(G) + o(k)\}$, where c_h and c'_k are constants depending, respectively, on the orientable and non-orientable genera of G .*

Notice that it is sufficient to consider connected graphs because the bondage number of a disconnected graph G is the minimum of the bondage numbers of its components.

In this paper, we provide constant upper bounds for the bondage number of graphs on topological surfaces, which can be used as the first estimation for the constants c_h and c'_k of Conjecture 5. Also, we improve upper bounds of Theorem 4, and show tight lower bounds for the number of vertices of graphs 2-cell embeddable on topological surfaces of a given genus. Also, we provide stronger upper bounds for graphs with no triangles and graphs with the number of vertices larger than a certain threshold in terms of the genera h and k . This provides ideas for improvements of our results in case of some restricted classes of graphs, shows that the bondage number is at most eleven and settles Teschner's Conjecture 1 in positive for almost all graphs.

2. Graphs on the topological surfaces

The planar graphs are precisely the graphs that can be drawn with no crossing edges on the sphere S_0 . A topological surface S can be obtained from the sphere S_0 by adding a number of handles or crosscaps. If we add h , $h \geq 1$, handles to S_0 , we obtain an orientable surface S_h , which is often referred to as the h -holed torus. The number h is called the *orientable genus* of S_h .

If we add k , $k \geq 1$, crosscaps to the sphere S_0 , we obtain a non-orientable surface N_k . The number k is called the *non-orientable genus* of N_k . Any topological surface is homeomorphically equivalent either to S_h ($h \geq 0$), or to N_k ($k \geq 1$). For example, S_1 , N_1 , N_2 are the *torus*, the *projective plane*, and the *Klein bottle*, respectively.

A graph G is *embeddable* on a topological surface S if it admits a drawing on the surface with no crossing edges. Such a drawing of G on the surface S is called an *embedding* of G on S . Notice that there can be many different embeddings of the same graph G on a particular surface S . The embeddings can be distinguished and classified by different properties. The set of faces of a particular embedding of G on S is denoted by $F(G)$, $|F(G)| = f$.

An embedding of G on the surface S is a *2-cell embedding* if each face of the embedding is homeomorphic to an open disk. In other words, a 2-cell embedding is an embedding on S that “fits” the surface. This is expressed in the Euler’s formula (2) of Theorem 6 below. For example, a cycle C_n ($n \geq 3$) does not have a 2-cell embedding on the torus, but it has 2-cell embeddings on the sphere and the projective plane. Similarly, a planar graph may have 2-cell and non-2-cell embeddings on the torus. An algorithm to transform a planar 2-cell embedding into a toroidal 2-cell embedding, whenever possible, can be found in Gagarin et al. [7], pp. 358–360. Similar algorithms to transform a 2-cell embedding of genus h , $h \geq 1$, (resp., k , $k \geq 1$) into a 2-cell embedding of genus $h + 1$ (resp., $k + 1$), whenever possible, can be devised for orientable (resp., non-orientable) surfaces by analogy, with more cases to consider. See also how to transform a planar 2-cell embedding of a graph with a cycle into a projective-planar 2-cell embedding in Kocay and Kreher [16], p. 364.

The following result is usually known as the (generalized) *Euler’s formula*. We state it here in a form similar to Thomassen [21].

Theorem 6 (Euler’s Formula, [21]). *Given a connected graph G with n vertices and m edges 2-cell embedded on a topological surface S ,*

$$n - m + f = \chi(S), \quad (2)$$

where either $\chi(S) = 2 - 2h$ and $S = S_h$, or $\chi(S) = 2 - k$ and $S = N_k$, and f is the number of faces of the 2-cell embedding on S .

Equation (2) is usually referred to as the *Euler’s formula* for an orientable surface S_h of genus h , $h \geq 0$, or a non-orientable surface N_k of genus k , $k \geq 1$, and the invariant $\chi(S)$ is the *Euler characteristic* of an orientable surface $S = S_h$ or a non-orientable surface $S = N_k$, respectively.

The *orientable genus* of a graph G is the smallest integer $h = h(G)$ such that G admits an embedding on an orientable topological surface S of genus h . The *non-orientable genus* of G is the smallest integer $k = k(G)$ such that G can be embedded on a non-orientable topological surface S of genus k . In general, $h(G) \neq k(G)$ (e.g., see [16], pp.367-368), and the embeddings on $S_{h(G)}$ and $N_{k(G)}$ must be 2-cell embeddings.

Lemma 7. *Given a graph G 2-cell embedded on an orientable surface S_h of genus h ,*

$$|V(G)| = n \geq \frac{3 + \sqrt{16h + 1}}{2} > 2\sqrt{h} + 1, \quad h \geq 1, \quad (3)$$

$|V(G)| = n \geq 1$ for $h = 0$, and, on a non-orientable surface N_k of genus k ,

$$|V(G)| = n \geq \frac{3 + \sqrt{8k + 1}}{2} > \sqrt{2k} + 1, \quad k \geq 1. \quad (4)$$

PROOF. From Euler's formula (2),

$$n - m + f = \chi(S).$$

Since $f \geq 1$, $m \leq \frac{n(n-1)}{2}$, we have

$$\chi(S) = n - m + f \geq n - \frac{n(n-1)}{2} + 1,$$

which gives

$$n^2 - 3n + 2(\chi(S) - 1) \geq 0.$$

Solving the corresponding quadratic equation for n ,

$$n = \frac{3 \pm \sqrt{17 - 8\chi(S)}}{2}. \quad (5)$$

Since n is a positive integer, plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (5) gives

$$n \geq \frac{3 + \sqrt{16h + 1}}{2} > 2\sqrt{h} + 1 \quad \text{for } h \geq 1,$$

$n \geq 1$ when $h = 0$, and

$$n \geq \frac{3 + \sqrt{8k + 1}}{2} > \sqrt{2k} + 1 \quad \text{for } k \geq 1,$$

respectively. □

The *maximum orientable* (resp., *non-orientable*) *genus* $h_M(G)$ (resp., $k_M(G)$) of a graph G is the largest integer h (resp., k) such that G has a 2-cell embedding on S_h (resp., N_k). The maximum genera of graphs are well-studied parameters (for example, see Huang [13] and Ringel [17]). Notice that, if $h(G)$ is the orientable genus of G , then G has 2-cell embeddings on the orientable surfaces of genus h , $h(G) \leq h \leq h_M(G)$. Similarly, G has 2-cell embeddings on the non-orientable surfaces of genus k , $k(G) \leq k \leq k_M(G)$.

The bounds of Lemma 7 are tight. Euler's formula (2) implies $h_M(G) \leq \lfloor \frac{m-n+1}{2} \rfloor$ and $k_M(G) \leq m - n + 1$, and 4-edge connected graphs are known to be *upper-embeddable*, i.e. to have $h_M(G) = \lfloor \frac{m-n+1}{2} \rfloor$ (e.g., see Jungerman [14]). Notice that complete graphs K_n , $n \geq 5$, are 4-edge connected, and $h(K_n) = \lceil \frac{(n-3)(n-4)}{12} \rceil$ (e.g., see [2], p. 281). Then,

for $h = 3$, Lemma 7 gives $n \geq 5$, and K_5 has $h(K_5) = 1$, $h_M(K_5) = 3$;
for $h = 5$, Lemma 7 gives $n \geq 6$, and K_6 has $h(K_6) = 1$, $h_M(K_6) = 5$;
for $h = 14$, Lemma 7 gives $n \geq 9$, and K_9 has $h(K_9) = 3$, $h_M(K_9) = 14$;
for $h = 18$, Lemma 7 gives $n \geq 10$, and K_{10} has $h(K_{10}) = 4$, $h_M(K_{10}) = 18$;
etc. In general, for $h = 4y^2 \pm y$, $y \in \mathbb{Z}^+$, the bound of Lemma 7 is tight. Notice that, from the proof of Lemma 7, the 2-cell embeddings of $K_{4y^2 \pm y}$ have a unique face on the surface of its maximum genus $h_M(K_{4y^2 \pm y})$, $y \in \mathbb{Z}^+$. Similar observations can be easily obtained for the non-orientable surfaces: a connected graph G which is not a tree has $k_M(G) = m - n + 1$ (see [17]).

A *triangle-free graph* G is a graph having no cycles of length 3. The lower bounds of Lemma 7 can be easily improved for graphs with no triangles as follows.

Corollary 8. *Given a triangle-free graph G 2-cell embedded on an orientable surface S_h of genus h , the number of vertices of G is*

$$n \geq 2(1 + \sqrt{2h}), \quad h \geq 1, \quad (6)$$

$n \geq 1$ when $h = 0$, and, on a non-orientable surface N_k of genus k ,

$$n \geq 2(1 + \sqrt{k}), \quad k \geq 1. \quad (7)$$

PROOF. The number of edges of a triangle-free graph G is $m \leq \frac{n^2}{4}$ (e.g., see [2], p. 45). The rest of the proof is similar to that of Lemma 7. \square

Notice that bipartite graphs are triangle-free, and all the results in this paper for the triangle-free graphs hold for the bipartite graphs as well.

3. Constant upper bounds for general graphs on the topological surfaces

Hartnell and Rall [10] proved the following.

Theorem 9 (Hartnell and Rall [10]). *The number of edges of a connected graph G with n vertices and the bondage number $b(G)$ is $m \geq \frac{n}{4}(b(G) + 1)$, and the bound is sharp.*

We use Theorem 9 to establish the following upper bounds on the bondage number of graphs.

Theorem 10. *Let G be a connected graph of the orientable genus $h = h(G)$, the non-orientable genus $k = k(G)$, and having $n = |V(G)|$ vertices. Then*

(i) $h = 0$ (G is planar) or $k = 1$ (G is projective-planar) implies $b(G) \leq 10$;

(ii) $h \geq 1$ and $n > 12(2h - 2)$, or $k \geq 2$ and $n > 12(k - 2)$, imply $b(G) \leq 11$;

(iii) $h \geq 2$ and $\frac{3+\sqrt{16h+1}}{2} \leq n \leq 12(2h-2)$ imply $b(G) \leq 11 + \frac{24(h-1)(3-\sqrt{16h+1})}{1-8h} = 11 + O(\sqrt{h})$, and $k \geq 3$ and $\frac{3+\sqrt{8k+1}}{2} \leq n \leq 12(k-2)$ imply $b(G) \leq 11 + \frac{12(k-2)(3-\sqrt{8k+1})}{1-4k} = 11 + O(\sqrt{k})$.

PROOF. As a corollary of Euler's formula (2), for $n \geq 3$, in general,

$$m \leq 3(n - \chi(S)) \quad (8)$$

(e.g., see [2], p. 279). Then, (8) and Theorem 9 give

$$\frac{n(b(G) + 1)}{4} \leq m \leq 3(n - \chi(S)),$$

which implies

$$b(G) \leq 11 - \frac{12\chi(S)}{n}. \quad (9)$$

Plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (9) gives

$$b(G) \leq 11 + \frac{12(2h - 2)}{n} \quad \text{and} \quad b(G) \leq 11 + \frac{12(k - 2)}{n},$$

respectively. The statements of Theorem 10 follow directly, applying the bounds (3) and (4) of Lemma 7 to obtain the statement of Theorem 10(iii). \square

Clearly, in the case of planar graphs, Theorem 3 provides a better upper bound, $b(G) \leq c_0 \leq 8$, than Theorem 10(i). Since there are no restrictions on the number of vertices in the cases of toroidal ($h = 1$), projective-planar ($k = 1$), and Klein bottle ($k = 2$) graphs in Theorem 10, we have the following general constant upper bounds.

Corollary 11. *For any connected projective-planar graph G , $b(G) \leq c'_1 \leq 10$, and any connected toroidal or Klein bottle graph G , $b(G) \leq 11$, i.e. $c_1 \leq 11$ and $c'_2 \leq 11$.*

The formulae of Theorem 10(iii) provide constant upper bounds for the surfaces of higher genera as follows.

Corollary 12. *For a connected graph G of orientable genus $h = h(G) \geq 2$ and non-orientable genus $k = k(G) \geq 3$, we have*

Orientable genus, h	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$b(G) \leq c_h \leq$	15	19	22	25	28	30	33	35	37	39	41	43	44	46
Non-orientable genus, k	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$b(G) \leq c'_k \leq$	13	15	17	19	21	22	24	25	27	28	29	30	32	33

Table 1: Constant upper bounds for the bondage number of graphs on topological surfaces of higher genera ($h \leq 15$ and $k \leq 16$).

Clearly, since only the direct arguments with Euler's formulae have been used in Theorem 10 and Lemma 7, the bounds for N_k , $k = 2h$, coincide with the corresponding upper bounds for S_h , $h \geq 1$: the Euler characteristics and corresponding Euler's formulae are the same in this case. However, surfaces S_h and N_{2h} , $h \geq 1$, are not equivalent, i.e. *non-homeomorphic* (e.g., see [21], pp.129–130), and the classes of graphs of orientable genus h and non-orientable genus $k = 2h$, $h \geq 1$, are quite different. Therefore, we conjecture that refinements of the results of Theorem 10 are going to provide different constant upper bounds for the bondage number of graphs embeddable on S_h and N_{2h} , $h \geq 1$.

4. Graphs with no triangles

The triangle-free graphs are exactly the graphs of girth at least 4. Fischermann et al. [6] have shown the following.

Theorem 13 (Fischermann et al. [6]). *A connected planar triangle-free graph G has $b(G) \leq 6$.*

We provide a simple proof of Theorem 13 and generalize it to all the other topological surfaces as follows.

Theorem 14. *Let G be a connected triangle-free graph of the orientable genus $h = h(G)$, the non-orientable genus $k = k(G)$, and having $n = |V(G)|$ vertices. Then*

- (i) $h = 0$ (G is planar) or $k = 1$ (G is projective-planar) implies $b(G) \leq 6$;
- (ii) $h \geq 1$ and $n > 8(2h - 2)$, or $k \geq 2$ and $n > 8(k - 2)$, imply $b(G) \leq 7$;
- (iii) $h \geq 2$ and $2(1 + \sqrt{2h}) \leq n \leq 8(2h - 2)$ imply $b(G) \leq 7 + \frac{8(h-1)}{1+\sqrt{2h}}$, and $k \geq 3$ and $2(1 + \sqrt{k}) \leq n \leq 8(k - 2)$ imply $b(G) \leq 7 + \frac{4(k-2)}{1+\sqrt{k}}$.

PROOF. In case of triangle-free graphs, $4f \leq 2m$ and $f \leq \frac{m}{2}$. Then, similarly to (8), as a corollary to Euler's formula (2), for $n \geq 3$, in general,

$$m \leq 2(n - \chi(S)) \quad (10)$$

Then, in this case, (10) and Theorem 9 give

$$\frac{n(b(G) + 1)}{4} \leq m \leq 2(n - \chi(S)),$$

which implies

$$b(G) \leq 7 - \frac{8\chi(S)}{n}. \quad (11)$$

Then, plugging in $\chi(S) = 2 - 2h$ and $\chi(S) = 2 - k$ into (11) gives

$$b(G) \leq 7 + \frac{8(2h - 2)}{n} \quad \text{and} \quad b(G) \leq 7 + \frac{8(k - 2)}{n},$$

respectively. The statements of Theorem 14 follow directly, applying the bounds (6) and (7) of Corollary 8 to Lemma 7 to obtain the statement of Theorem 14(iii). \square

Notice that, in the case of planar graphs, Theorem 14(i) provides the same upper bound, $b(G) \leq 6$, as the previously known upper bound of Theorem 13. Conclusions similar to Corollaries 11 and 12 with constant upper bounds for the bondage number of triangle-free graphs on topological surfaces can be drawn from Theorem 14 as well. In general, Theorem 14 provides stronger bounds than Theorem 10 in the case of triangle-free graphs.

5. Improved upper bounds in terms of the maximum vertex degree and the genera

In [8], we indicate that the upper bound (1) can be improved for bigger values of the genera h and k by adjusting the proofs of the corresponding theorems. This is because the assumptions $\delta(G) \geq h + 4$, $\delta(G) \geq k + 3$, corresponding arguments and reasoning, used respectively in the proofs of Theorems 8 and 9 in [8], become vacuous for bigger values of the genera h and k in view of the natural upper bounds on the minimum vertex degree: $\delta(G) \leq \lfloor \frac{5+\sqrt{1+48h}}{2} \rfloor$ for $h \geq 1$, $\delta(G) \leq \lfloor \frac{5+\sqrt{1+24k}}{2} \rfloor$ for $k \geq 2$ (e.g., see Sachs [18]), and $\delta(G) \leq 5$ for a planar or projective-planar graph, i.e. when $h = 0$ or $k = 1$. The suggested improvements are of the form

$$b(G) \leq \min\{\Delta(G) + h - a, \Delta(G) + k - b\}, \quad (12)$$

where $a \geq -1$, $b \geq 0$, $a, b \in \mathbb{Z}$, $h \geq t_a$, $k \geq t_b$, $t_a, t_b \in \mathbb{Z}^+$, and the thresholds t_a and t_b can be found only after going through the corresponding adjusted proofs. The upper bounds obtained by the adjustments are better than the following upper bound which is asymptotically better than the original bound (1):

$$b(G) \leq \min\{\Delta(G) + \lfloor \frac{3 + \sqrt{1 + 48h}}{2} \rfloor, \Delta(G) + \lfloor \frac{3 + \sqrt{1 + 24k}}{2} \rfloor\}, \quad (13)$$

$h \geq 1$ and $k \geq 1$. Bound (13) can be seen as a simple corollary to Lemma 15 below and the upper bounds on the minimum vertex degree in terms of the graph genera.

First, let us show where bounds (12) can be at least as good as bound (13). In other words, having fixed $a \geq -1$ and $b \geq 0$, $a, b \in \mathbb{Z}$, for which values of h and k can we have $h - a \leq \lfloor \frac{3+\sqrt{1+48h}}{2} \rfloor$ and $k - b \leq \lfloor \frac{3+\sqrt{1+24k}}{2} \rfloor$, respectively? Solving corresponding quadratic inequalities gives $h \in [a + \frac{15}{2} - \frac{\sqrt{48a+217}}{2}, a + \frac{15}{2} + \frac{\sqrt{48a+217}}{2}]$ and $k \in [b + \frac{9}{2} - \frac{\sqrt{24b+73}}{2}, b + \frac{9}{2} + \frac{\sqrt{24b+73}}{2}]$. Then, clearly, thresholds t_a and t_b computed in the adjusted proofs of Theorems 8 and 9 in [8], resp., must be in the same intervals to guarantee the bounds of the form (12) make sense. For example, when $b = 5$ in (12), the genus $k \in [3, 16]$ guarantee that (12) is at least as good as (13), and the threshold is $t_b = 13$, i.e. $b(G) \leq \Delta(G) + k - 5$ for $k \geq 13$.

Apparently, the implicit improvements of the form (12) suggested in [8] (or, maybe, even better) are explicitly obtained and presented in Huang [12]. In this section, we improve the upper bounds of [8] and [12] as follows.

One of the classical upper bounds on the bondage number can be stated as follows.

Lemma 15 (Hartnell and Rall [9]). *For any edge uv in a graph G , we have $b(G) \leq d(u) + d(v) - 1 - d_{uv}$, where $d_{uv} = |N(u) \cap N(v)|$. In particular, this implies that $b(G) \leq \Delta(G) + \delta(G) - 1$ (see also [1, 5]).*

Having a graph G embedded on a surface S , each edge $e_i = uv \in E(G)$, $i = 1, \dots, m$, is assigned two weights, $w_i = \frac{1}{d(u)} + \frac{1}{d(v)}$ and $f_i = \frac{1}{m'} + \frac{1}{m''}$, where m' is the number of edges on the boundary of a face on one side of e_i , and m'' is the number of edges on the boundary of the face on the other side of e_i . Notice that, in an embedding on a surface, an edge e_i may be not separating two distinct faces, but instead can appear twice on the boundary of the same face, and, in this case, $f_i = \frac{2}{m'} = \frac{2}{m''}$.

We have

$$\sum_{i=1}^m w_i = n, \quad \sum_{i=1}^m f_i = f.$$

and, by Euler's formula (2),

$$\sum_{i=1}^m (w_i + f_i - 1) = n + f - m = \chi(S),$$

or, in other words,

$$\sum_{i=1}^m \left(w_i + f_i - 1 - \frac{\chi(S)}{m} \right) = 0. \quad (14)$$

Now, each edge $e_i = uv \in E(G)$, $i = 1, \dots, m$, can be associated with the weight $Q(e_i) = w_i + f_i - 1 - \frac{\chi(S)}{m}$ called, depending on S , the *oriented* or *non-oriented curvature* of the edge, respectively.

Theorem 16. *Let G be a connected graph 2-cell embeddable on an orientable surface of genus $h \geq 0$. Then*

$$b(G) \leq \begin{cases} \Delta(G) + \lceil h^{0.7} \rceil + 2, & \text{for } h \leq 5, \\ \Delta(G) + \lceil h^{0.7} \rceil + 3, & \text{for } h \geq 6. \end{cases} \quad (15)$$

PROOF. Suppose G is 2-cell embedded on the h -holed torus S_h . Denote by

$$\tau = \begin{cases} \lceil h^{0.7} \rceil - 1, & \text{for } h \leq 5, \\ \lceil h^{0.7} \rceil, & \text{for } h \geq 6. \end{cases}$$

Then we have to prove

$$b(G) \leq \Delta(G) + \tau + 3.$$

If $\delta(G) \leq \tau + 4$, then, by Lemma 15,

$$b(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + \tau + 3,$$

as required, and inequality (15) holds.

Therefore, we can assume $\delta(G) \geq \tau + 5$. Suppose the opposite, $b(G) \geq \Delta(G) + \tau + 4$. Then, by Lemma 15, for any edge $e_i = uv$, $i = 1, \dots, m$, we have

$$d(u) + d(v) - 1 - d_{uv} \geq b(G) \geq \Delta(G) + \tau + 4,$$

which implies

$$d(u) + d(v) \geq \Delta(G) + \tau + 5 + d_{uv}, \quad (16)$$

and $\Delta(G) \geq d(u)$, $\Delta(G) \geq d(v)$. Without loss of generality, assume $d(u) \leq d(v)$. Then we have the following three cases.

Case 1: $d(u) = \tau + 5$. By (16), $d(v) \geq \Delta(G) + d_{uv}$, which implies $d(v) = \Delta(G)$ and $d_{uv} = 0$. Therefore, $m' \geq 4$ and $m'' \geq 4$ in this case, and $Q(e_i) = w_i + f_i - 1 + \frac{2h-2}{m}$. When $h = 0$, $\tau = -1$, and

$$Q(e_i) \leq \frac{2}{4} - \frac{1}{2} - \frac{2}{m} < 0.$$

When $h \geq 1$, $d_{uv} = 0$ implies $n \geq d(u) + d(v) \geq 2\tau + 10$, and $m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau+10)(\tau+5)}{2} = (\tau+5)^2$, and

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{(\tau+5)^2} < 0.$$

Case 2: $d(u) = \tau + 6$. Then $d(v) \geq \tau + 6$. By (16), $d(v) \geq \Delta(G) - 1 + d_{uv}$. If $d_{uv} \geq 2$, then $d(v) \geq \Delta(G) + 1$, a contradiction. Therefore, $d_{uv} \leq 1$

and, without loss of generality, $m' \geq 3$ and $m'' \geq 4$ in this case. We have $Q(e_i) \leq \frac{2}{\tau+6} + \frac{1}{3} + \frac{1}{4} - 1 + \frac{2h-2}{m} = \frac{2}{\tau+6} - \frac{5}{12} + \frac{2h-2}{m}$. When $h = 0$, $\tau = -1$, and

$$Q(e_i) \leq \frac{2}{5} - \frac{5}{12} - \frac{2}{m} = -\frac{1}{60} - \frac{2}{m} < 0.$$

When $h \geq 1$, $d_{uv} \leq 1$ implies $n \geq d(u) + d(v) - 1 \geq 2\tau + 11$, and $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(2\tau+9)(\tau+5)}{2} = \frac{2\tau^2+21\tau+57}{2}$, and

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2\tau^2+21\tau+57} < 0.$$

Case 3: $d(u) \geq \tau + 7$. Then $d(v) \geq \tau + 7$, and $m' \geq 3$, $m'' \geq 3$. We have $Q(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{m}$. When $h = 0$, $\tau = -1$, and

$$Q(e_i) \leq \frac{2}{6} - \frac{1}{3} - \frac{2}{m} < 0.$$

When $h \geq 1$, inequality (16) implies $n \geq d(u) + d(v) - d_{uv} \geq \Delta(G) + \tau + 5 \geq 2\tau + 12$, then $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+14+(2\tau+10)(\tau+5)}{2} = \tau + 7 + (\tau + 5)^2 = \tau^2 + 11\tau + 32$, and

$$Q(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{\tau^2+11\tau+32} < 0 \quad \text{for } h \in \mathbb{Z}^+.$$

Thus, $Q(e_i) < 0$, $i = 1, \dots, m$, for any $h \geq 0$, $h \in \mathbb{Z}$, and $\sum_{i=1}^m Q(e_i) < 0$, a contradiction with (14). Therefore $b(G) \leq \Delta(G) + \tau + 3$, as required. \square

If the genus h is fixed, by excluding a finite number of graphs, we can improve the bound of Theorem 16 as follows.

Corollary 17. *For a connected graph G 2-cell embeddable on an orientable surface of genus $h \geq 1$, we have:*

- (a) $b(G) \leq \Delta(G) + \lceil \ln^2 h \rceil + 3$ if $n \geq h$;
- (b) $b(G) \leq \Delta(G) + \lceil \ln h \rceil + 3$ if $n \geq h^{1.9}$;
- (c) $b(G) \leq \Delta(G) + 4$ if $n \geq h^{2.5}$.

PROOF. Denote by:

- (a) $\tau = \lceil \ln^2 h \rceil$;
- (b) $\tau = \lceil \ln h \rceil$;

(c) $\tau = 1$.

Then, as in Theorem 16, we have to prove

$$b(G) \leq \Delta(G) + \tau + 3.$$

If $\delta(G) \leq \tau + 4$, then, by Lemma 15,

$$b(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + \tau + 3,$$

as required.

Therefore, as in Theorem 16, we can assume $\delta(G) \geq \tau + 5$, suppose the opposite, $b(G) \geq \Delta(G) + \tau + 4$, and, by Lemma 15, for any edge $e_i = uv$, $i = 1, \dots, m$, the inequality (16) holds here as well. Again, without loss of generality, $d(u) \leq d(v)$. Then we have the following three cases.

Case 1: $d(u) = \tau + 5$. By (16), $d(v) \geq \Delta(G) + d_{uv}$, $d(v) = \Delta(G)$, and $d_{uv} = 0$. Therefore, $m' \geq 4$ and $m'' \geq 4$. Then $Q(e_i) = w_i + f_i - 1 + \frac{2h-2}{m} \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{m}$, and $m \geq \frac{n\delta(G)}{2} \geq \frac{h^x(\tau+5)}{2}$ and, similarly to the proof of Theorem 16, $m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau+10)(\tau+5)}{2} = (\tau+5)^2$. This implies, respectively,

$$Q(e_i) = Q'(e_i) \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2(2h-2)}{h^x(\tau+5)}$$

and

$$Q(e_i) = Q''(e_i) \leq \frac{2}{\tau+5} - \frac{1}{2} + \frac{2h-2}{(\tau+5)^2}.$$

We have:

- (a) $x = 1$, then $Q'(e_i) < 0$ if $h \geq 12$, and $Q''(e_i) < 0$ if $h \leq 11$;
- (b) $x = 1.9$, then $Q'(e_i) < 0$ if $h \neq 2$, and $Q''(e_i) < 0$ if $h = 2$;
- (c) $x = 2.5$, then $Q'(e_i) < 0$, $h \geq 1$.

Case 2: $d(u) = \tau + 6$. Then, similarly to the proof of Theorem 16, $d(v) \geq \tau + 6$, and, by (16), $d(v) \geq \Delta(G) - 1 + d_{uv}$, $d_{uv} \leq 1$, and, without loss of generality, $m' \geq 3$ and $m'' \geq 4$. Then $Q(e_i) \leq \frac{2}{\tau+6} - \frac{5}{12} + \frac{2h-2}{m}$, and $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(n-2)(\tau+5)}{2} = \frac{2+n(\tau+5)}{2} \geq \frac{2+h^x(\tau+5)}{2}$ and, similarly to the proof of Theorem 16, $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(2\tau+9)(\tau+5)}{2} = \frac{2\tau^2+21\tau+57}{2}$. This implies, respectively,

$$Q(e_i) = Q'(e_i) \leq \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2+h^x(\tau+5)}$$

and

$$Q(e_i) = Q''(e_i) \leq \frac{2}{\tau+6} - \frac{5}{12} + \frac{2(2h-2)}{2\tau^2+21\tau+57}.$$

We have:

- (a) $x = 1$, then $Q'(e_i) < 0$ if $h \geq 17$, and $Q''(e_i) < 0$ if $h \leq 16$;
- (b) $x = 1.9$, then $Q'(e_i) < 0$ if $h \neq 2$, and $Q''(e_i) < 0$ if $h = 2$;
- (c) $x = 2.5$, then $Q'(e_i) < 0$, $h \geq 1$.

Case 3: $d(u) \geq \tau+7$. Then we have $d(v) \geq \tau+7$, $m' \geq 3$, $m'' \geq 3$, $Q(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{m}$, and $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+14+(n-2)(\tau+5)}{2} = \frac{4+n(\tau+5)}{2} \geq \frac{4+h^x(\tau+5)}{2}$ and, similarly to the proof of Theorem 16, $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{4+n(\tau+5)}{2} \geq \frac{4+(2\tau+12)(\tau+5)}{2} = 2 + (\tau+6)(\tau+5) = \tau^2 + 11\tau + 32$. This implies, respectively,

$$Q(e_i) = Q'(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2(2h-2)}{4+h^x(\tau+5)}$$

and

$$Q(e_i) = Q''(e_i) \leq \frac{2}{\tau+7} - \frac{1}{3} + \frac{2h-2}{\tau^2+11\tau+32}.$$

We have:

- (a) $x = 1$, then $Q'(e_i) < 0$ if $h \geq 28$, and $Q''(e_i) < 0$ if $h \leq 27$;
- (b) $x = 1.9$, then $Q'(e_i) < 0$ if $h \geq 5$, and $Q''(e_i) < 0$ if $h \leq 4$;
- (c) $x = 2.5$, then $Q'(e_i) < 0$ if $h \neq 2$, and $Q''(e_i) < 0$ if $h = 2$.

Thus, $Q(e_i) < 0$, $i = 1, \dots, m$, for any $h \geq 1$, $h \in \mathbb{Z}$, and $\sum_{i=1}^m Q(e_i) < 0$, a contradiction with (14). Therefore $b(G) \leq \Delta(G) + \tau + 3$, as required. \square

Theorem 18. *Let G be a connected graph 2-cell embeddable on a non-orientable surface of genus $k \geq 1$. Then*

$$b(G) \leq \begin{cases} \Delta(G) + \lceil k^{0.6} \rceil + 1, & \text{for } k \leq 5, \\ \Delta(G) + \lceil k^{0.6} \rceil + 2, & \text{for } k \geq 6. \end{cases} \quad (17)$$

PROOF. The proof is similar to that of Theorem 16 above and goes as follows. Let G be 2-cell embedded on N_k . Denote by

$$\tau = \begin{cases} \lceil k^{0.6} \rceil - 1, & \text{for } k \leq 5, \\ \lceil k^{0.6} \rceil, & \text{for } k \geq 6. \end{cases}$$

Then we have to prove that

$$b(G) \leq \Delta(G) + \tau + 2.$$

If $\delta(G) \leq \tau + 3$, then, by Lemma 15,

$$b(G) \leq \Delta(G) + \delta(G) - 1 \leq \Delta(G) + \tau + 2,$$

as required, and inequality (17) holds.

Therefore, assume $\delta(G) \geq \tau + 4$. Suppose that $b(G) \geq \Delta(G) + \tau + 3$. Then, by Lemma 15, for any edge $e_i = uv$, $i = 1, \dots, m$,

$$d(u) + d(v) - 1 - d_{uv} \geq b(G) \geq \Delta(G) + \tau + 3,$$

i.e.

$$d(u) + d(v) \geq \Delta(G) + \tau + 4 + d_{uv}. \quad (18)$$

Without loss of generality, $d(u) \leq d(v)$. We have the following three cases.

Case 1: $d(u) = \tau + 4$. By (18), $d(v) \geq \Delta(G) + d_{uv}$, which implies $d(v) = \Delta(G)$, $d_{uv} = 0$, and $m' \geq 4$, $m'' \geq 4$. In this case, we have $Q(e_i) = w_i + f_i - 1 + \frac{k-2}{m} \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{m}$. When $k = 1$,

$$Q(e_i) \leq \frac{2}{4} - \frac{1}{2} - \frac{1}{m} < 0.$$

When $k \geq 2$, $d_{uv} = 0$ implies $n \geq d(u) + d(v) \geq 2\tau + 8$, and $m \geq \frac{n\delta(G)}{2} \geq \frac{(2\tau+8)(\tau+4)}{2} = (\tau+4)^2$, and

$$Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{(\tau+4)^2} < 0.$$

Case 2: $d(u) = \tau + 5$. Then $d(v) \geq \tau + 5$. By (18), $d(v) \geq \Delta(G) - 1 + d_{uv}$, which implies $d_{uv} \leq 1$, and, without loss of generality, $m' \geq 3$ and $m'' \geq 4$ in this case. Then we have $Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{m}$. When $k = 1$,

$$Q(e_i) \leq \frac{2}{5} - \frac{5}{12} - \frac{1}{m} = -\frac{1}{60} - \frac{1}{m} < 0.$$

When $k \geq 2$, $n \geq d(u) + d(v) - 1 \geq 2\tau + 9$, and $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+10+(2\tau+7)(\tau+4)}{2} = \frac{2\tau^2+17\tau+38}{2} = \tau^2 + 8.5\tau + 19$, and

$$Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{\tau^2 + 8.5\tau + 19} < 0.$$

Case 3: $d(u) \geq \tau + 6$. Then $d(v) \geq \tau + 6$, and $m' \geq 3$, $m'' \geq 3$. We have $Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{m}$. When $k = 1$,

$$Q(e_i) \leq -\frac{1}{m} < 0.$$

When $k \geq 2$, inequality (18) implies $n \geq d(u) + d(v) - d_{uv} \geq \Delta(G) + \tau + 4 \geq 2\tau + 10$. Then $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(2\tau+8)(\tau+4)}{2} = \tau + 6 + (\tau + 4)^2 = \tau^2 + 9\tau + 22$, and

$$Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{\tau^2+9\tau+22} < 0 \quad \text{for } k \in \mathbb{Z}.$$

Thus, $Q(e_i) < 0$, $i = 1, \dots, m$, for any $k \geq 1$, $k \in \mathbb{Z}$, and $\sum_{i=1}^m Q(e_i) < 0$, a contradiction with (14). Therefore $b(G) \leq \Delta(G) + \tau + 2$, as required. \square

The bounds of Theorem 18 can be improved if the number of vertices n is restricted from below by some function of the non-orientable genus k . If k is fixed, then only a finite number of graphs is forbidden, i.e. the following bounds are true for almost all graphs.

Corollary 19. *Let G be a connected graph 2-cell embeddable on a non-orientable surface of genus $k \geq 1$. Then:*

- (a) $b(G) \leq \Delta(G) + \lceil \ln^2 k \rceil + 2$ if $n \geq k/6$;
- (b) $b(G) \leq \Delta(G) + \lceil \ln k \rceil + 2$ if $n \geq k^{1.6}$;
- (c) $b(G) \leq \Delta(G) + 3$ if $n > k^2$.

PROOF. For $k = 1$, the result follows from Theorem 18. Therefore, consider $k \geq 2$. Denote by:

- (a) $\tau = \lceil \ln^2 k \rceil$;
- (b) $\tau = \lceil \ln k \rceil$;
- (c) $\tau = 1$.

In each case, similarly to the proof of Theorem 18, we have to prove

$$b(G) \leq \Delta(G) + \tau + 2.$$

Again, similarly to the proof of Theorem 18, we can assume $\delta(G) \geq \tau + 4$, and suppose the opposite, $b(G) \geq \Delta(G) + \tau + 3$. Then, by Lemma 15, for any edge $e_i = uv$, $i = 1, \dots, m$, the inequality (18) holds in this new context as well. Again, without loss of generality, $d(u) \leq d(v)$.

- Case 1:* $d(v) \geq d(u) = \tau + 4$. We have $Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{m}$.
- (a) $m \geq \frac{n\delta(G)}{2} \geq \frac{k(\tau+4)}{12}$, then $Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{12(k-2)}{k(\tau+4)} < 0$ if $k \geq 122$, and, similarly to the proof of Theorem 18, $Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{k-2}{(\tau+4)^2} < 0$ if $k \leq 121$.
- (b) $m \geq \frac{n\delta(G)}{2} \geq \frac{k^{1.6}(\tau+4)}{2}$, then $Q(e_i) \leq \frac{2}{\tau+4} - \frac{1}{2} + \frac{2(k-2)}{k^{1.6}(\tau+4)} < 0$ for $k \geq 2$.
- (c) $m \geq \frac{n\delta(G)}{2} > \frac{k^2(\tau+4)}{2}$, then $Q(e_i) < \frac{2}{\tau+4} - \frac{1}{2} + \frac{2(k-2)}{k^2(\tau+4)} < 0$ for $k \geq 2$.

- Case 2:* $d(v) \geq d(u) = \tau + 5$, $Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{m}$.
- (a) $m \geq \frac{k(\tau+4)}{12}$, then $Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{12(k-2)}{k(\tau+4)} < 0$ if $k \geq 219$, and, similarly to the proof of Theorem 18, $Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{k-2}{\tau^2+8.5\tau+19} < 0$ if $k \leq 218$.
- (b) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+10+(k^{1.6}-2)(\tau+4)}{2} = \frac{k^{1.6}(\tau+4)+2}{2}$, then $Q(e_i) \leq \frac{2}{\tau+5} - \frac{5}{12} + \frac{2(k-2)}{k^{1.6}(\tau+4)+2} < 0$ for $k \geq 2$.
- (c) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} > \frac{2\tau+10+(k^2-2)(\tau+4)}{2} = \frac{k^2(\tau+4)+2}{2}$, then $Q(e_i) < \frac{2}{\tau+5} - \frac{5}{12} + \frac{2(k-2)}{k^2(\tau+4)+2} < 0$ for $k \geq 2$.

- Case 3:* $d(v) \geq d(u) = \tau + 6$, $Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{m}$.
- (a) $m \geq \frac{k(\tau+4)}{12}$, then $Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{12(k-2)}{k(\tau+4)} < 0$ if $k \geq 439$, and, similarly to the proof of Theorem 18, $Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{k-2}{\tau^2+9\tau+22} < 0$ if $k \leq 438$.
- (b) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} \geq \frac{2\tau+12+(k^{1.6}-2)(\tau+4)}{2} = \frac{k^{1.6}(\tau+4)+4}{2}$, then $Q(e_i) \leq \frac{2}{\tau+6} - \frac{1}{3} + \frac{2(k-2)}{k^{1.6}(\tau+4)+4} < 0$ for $k \geq 2$, $k \in \mathbb{Z}$.
- (c) $m \geq \frac{d(u)+d(v)+(n-2)\delta(G)}{2} > \frac{2\tau+12+(k^2-2)(\tau+4)}{2} = \frac{k^2(\tau+4)+4}{2}$, then $Q(e_i) < \frac{2}{\tau+6} - \frac{1}{3} + \frac{2(k-2)}{k^2(\tau+4)+4} < 0$ for $k \geq 2$.

Thus, $Q(e_i) < 0$, $i = 1, \dots, m$, for any $k \geq 2$, $k \in \mathbb{Z}$, and $\sum_{i=1}^m Q(e_i) < 0$, a contradiction with (14). Therefore $b(G) \leq \Delta(G) + \tau + 2$, as required. \square

6. Teschner's conjecture and final remarks

Combining Corollary 11 with the results of [8] and Theorem 6 in [3], we obtain a statements similar to Theorem 3.

Theorem 20. *For any connected projective-planar graph G ,*

$$b(G) \leq \min\{10, \Delta(G) + 2\},$$

and any connected toroidal or Klein bottle graph G ,

$$b(G) \leq \min\{11, \Delta(G) + 3\}.$$

Theorem 20 and Theorem 3 settle Teschner's Conjecture 1 in positive for all the planar and projective-planar graphs G with $\Delta(G) \geq 4$, and all the toroidal and Klein bottle graphs G with $\Delta(G) \geq 6$, respectively. Theorem 10(ii), Conjecture 17(c), and Conjecture 19(c) settle Teschner's Conjecture 1 in positive for almost all graphs G with $\Delta(G) \geq 6$ or $\Delta(G) \geq 8$, depending on the number of vertices n in terms of the genera h and k of G .

Since, for each particular surface S of orientable genus h or non-orientable genus k , the number of graphs embeddable in S is infinite, and the number of graphs not satisfying Theorem 10(ii), Conjecture 17(c), Conjecture 19(c), Theorem 20 or Theorem 3, and having the maximum vertex degree $\Delta(G)$ bounded by 8, 6, or 4, respectively, is finite, we can conclude that Teschner's Conjecture 1 holds for almost all graphs in general. Also, this provides particular cases where Teschner's Conjecture 1 must be verified to be settled for all the graphs in general. We hope the hierarchies of upper bounds by the graph genera are going to be useful to solve Conjecture 1 for all the graphs.

In view of the results of [12] and upper bounds (13), the bounds of Theorems 16 and 18 are asymptotically not tight for larger values of the genera $h = h(G)$ and $k = k(G)$. However, Theorems 16 and 18 provide better results than the upper bounds of [12] for values of the genera h and k raging up to several hundreds. Therefore, we would suggest searching for general upper bounds of the form $\Delta(G) + O(\sqrt{h})$ and $\Delta(G) + O(\sqrt{k})$ by combining or improving the results of Theorems 16 and 18 with those of [12].

Computations in the proofs of Theorems 16 and 18 and Corollaries 17 and 19 are done by using software Maple 9.5 and MS Excel. Also, in general, as presented, these four proofs rely on assumptions of appropriate asymptotic behaviour of functions. It would be interesting to obtain similar proofs without using computational tools or the assumptions. However, since asymptotically the upper bounds (13) and corresponding upper bounds of [12] are stronger, one needs to do the computations only for a finite number of cases, and there is no need to justify the asymptotic behaviour of functions in general.

For the constant upper bounds for topological surfaces of Theorems 10 and 3 (see also Corollaries 11 and 12), we would suggest to refine them to obtain tight constant upper bounds. For example, Fischermann et al. [6] asked whether there exist planar graphs of bondage numbers 6, 7, or 8. A class of planar graphs with the bondage number equal to 6 is shown in [3], and $6 \leq c_0 \leq 8$ in the case of planar graphs. The next surfaces to consider should be the torus S_1 , projective plane N_1 , and Klein bottle N_2 . Therefore, we would suggest to try to improve, if possible, the results of Corollary 11

(see also Theorem 20).

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